Determinants and Inverses of Matrices

only square matrices

Definition of the Determinant of a 2 x 2 Matrix

The determinant of the matrix $A=\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ is $\det(A)=|A|=\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}=a_1b_2-a_2b_1$

Example 1 – Find the determinant of each matrix

a. $A=\begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$

$\det(A)=|A|=2\cdot 2-1\cdot -3$

$|A|=7$

b. $B=\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$

$\det(B)=|B|=2\cdot 2-4\cdot 1$

$|B|=0$

c. $C=\begin{bmatrix} 0 & \frac{3}{2} \\ 2 & 4 \end{bmatrix}$

$\det(C)=|C|=0\cdot 4-2\cdot \frac{3}{2}$

$|C|=-3$

The Determinant of 3 x 3 Matrix

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}=\left|\begin{array}{ccc|cc} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{array}\right.=(aei+bfg+cdh)-(gec+hfa+idb)$$

Example 2 – Find the Determinant of each Matrix

a. $\left[\begin{array}{ccc} 5 & -8 & 4 \\ 4 & 2 & 1 \\ 1 & 1 & -5 \end{array}\right]\begin{array}{cc} 5 & -8 \\ 4 & 2 \\ 1 & 1 \end{array}$

$(5\cdot 2\cdot -5+-8\cdot 1\cdot 1+4\cdot 4\cdot 1)-(1\cdot 2\cdot 4+1\cdot 1\cdot 5+-5\cdot 4\cdot -8)$

$(-50+-8+16)-(8+5+160)$

$(-42)-(173)$

$\det=-215$

b. $\left[\begin{array}{ccc} 10 & 7 & -8 \\ 4 & 2 & 5 \\ 3 & -2 & -5 \end{array}\right]\begin{array}{cc} 10 & 7 \\ 4 & 2 \\ 3 & -2 \end{array}$

$(-100+105+64)-(-48+-100+-140)$

$69--288$

$\det=357$

Square matrices only I: Identity Matrix

$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Definition of the Inverse of a Square Matrix

Let A be an n x n matrix. If there exists a matrix A^{-1} such that $AA^{-1} = I_n = A^{-1}A$ then A^{-1} is called the inverse of A. The symbol A^{-1} is read "A inverse"

The Inverse of a Matrix

Example 1 - Show that B is the inverse of A, where $A = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$

$$A \cdot B = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1+2 & 2+-2 \\ -1+1 & 2+-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \checkmark$$

$$B \cdot A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1+2 & 2+-2 \\ -1+1 & 2+-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \checkmark$$

Example 2 – Show the B is the inverse of A, where $A = \begin{bmatrix} 2 & -17 & 11 \\ -1 & 11 & -7 \\ 0 & 3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix}$

Show $A \cdot B = I$
and $B \cdot A = I$
in your calculator

Finding the Inverse of a 2 x 2 Matrix

det ≠ 0

If A is a 2 x 2 matrix, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then A is invertable if and only if ad – bc ≠ 0. If ad – cb ≠ 0, the inverse is $A^{-1} = \dfrac{1}{ad - cb}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Example 3 – If possible, find the inverse of the matrix.

a. $A = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$

$$A^{-1} = \frac{1}{3 \cdot 2 - (-2 \cdot -1)}\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1/2 & 1/4 \\ 1/2 & 3/4 \end{bmatrix}$$

b. $B = \begin{bmatrix} 3 & -1 \\ -6 & 2 \end{bmatrix}$

$$B^{-1} = \frac{1}{6-6}\begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}$$

$$B^{-1} = \frac{1}{0}\begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}$$

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*The inverse DNE because det = 0

Finding the Inverse of a 3 x 3 Matrix

Example 5 - If possible, using a calculator find the inverse of the $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix}$

$$A^{-1} = \begin{bmatrix} -2 & -3 & 1 \\ -3 & -3 & 1 \\ -2 & -4 & 1 \end{bmatrix}$$